Essential Understanding Series

Developing Essential Understanding *of* Ratios, Proportions, *and* Proportional Reasoning *for* Teaching Mathematics *in* Grades 6–8

Joanne Lobato
San Diego State University
San Diego, California

Amy B. Ellis
University of Wisconsin–Madison
Madison, Wisconsin

Randall I. Charles
Volume Editor
Carmel, California

Rose Mary Zbiek
Series Editor
The Pennsylvania State University
University Park, Pennsylvania

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Essential Understanding 4

A number of mathematical connections link ratios and fractions:

- *Ratios are often expressed in fraction notation, although ratios and fractions do not have identical meaning.*
- *Ratios are often used to make "part-part" comparisons, but fractions are not.*
- *Ratios and fractions can be thought of as overlapping sets.*
- *Ratios can often be meaningfully reinterpreted as fractions.*

Interpreting a fraction as a ratio is one of several interpretations of fractions discussed in Developing Essential Understanding of Rational Numbers for Teaching Mathematics in Grades 3–5 *(Barnett-Clarke et al., forthcoming).*

Because ratios can be written in fraction form as $^{a}/_{b}$, many students believe that *ratio* is just another word for *fraction*. The use of fraction language in discussions of problems involving ratios can be particularly confusing to students. For example, in discussing the solution to the proportion shown in connection with the problem in figure 1.17, a teacher may say, "Six is the answer because $^{2}/_{3}$ and $^{6}/_{9}$ are equivalent fractions." Essential Understanding 4 highlights the mathematical connections between ratios and fractions. The notation $^{a}/_{b}$ can easily cloud students' understanding of ratios if the students have not yet grasped the connections between ratios and fractions.

> If you make orange juice in the ratio of 2 cans of orange concentrate to 3 cans of water, how many cans of orange concentrate do you need to use with 9 cans of water?
>
> $$\frac{2}{3} = \frac{x}{9}$$

Fig. 1.17. A typical textbook problem expressed as a proportion

Ratios and fractions do not have identical meanings. Ratios are often used to make part-part comparisons, though fractions are not. For example, consider a salad dressing that is 2 parts vinegar to 5 parts oil. The *ratio* of vinegar to oil is expressed as 2 : 5, 2 to 5, or $^{2}/_{5}$. In this context, $^{2}/_{5}$ is a part-part comparison. In contrast, the *fraction* of the salad dressing that is oil is $^{5}/_{7}$, which is a part-whole comparison, and the fraction that is vinegar is $^{2}/_{7}$, which is another part-whole comparison.

Ratios and fractions can be conceived as overlapping sets (Clark, Berenson, and Cavey 2003). An example of a ratio that is not a fraction is the *golden ratio*

$$\left(\frac{\sqrt{5}+1}{2}\right).$$

This ratio is an irrational number, whereas fractions are rational numbers. A second example of a ratio that is not a fraction is the part-part comparison of vinegar to oil presented above—namely, $^{2}/_{5}$. Furthermore, ratios can involve more than two terms, such as the ratio of numbers of containers of whole milk to numbers of containers of low-fat milk to numbers of containers of nonfat milk in a certain store (e.g., 5 : 3 : 1). In the intersection of the sets of ratios and fractions are ratios that are formed as part-whole comparisons, as illustrated in figure 1.18. For example, the ratio of vinegar to total ingredients in the salad dressing—namely, 2 : 7—can also be thought of as a fraction: two-sevenths of the dressing is vinegar.

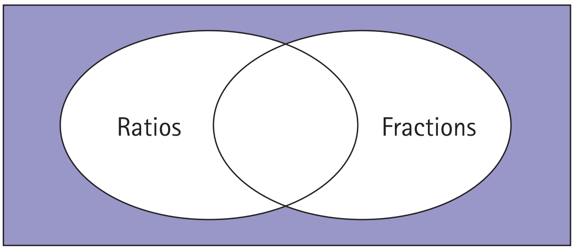

Fig. 1.18. Ratios and fractions as overlapping sets

At the other extreme are the various ways of thinking of fractions as entities other than part-whole comparisons. These ways include thinking of a fraction as a point on a number line (e.g., $^{8}/_{9}$ as a number between 0 and 1 on a number line). A fraction conceived in this way is often called a "fraction-as-measure." A fraction can also be thought of as an operator, such as a "shrinker" or "stretcher," which transforms the size of a given amount. Consider, for example, shrinking an amount by the fraction $^{1}/_{3}$. In neither case—fraction as measure or fraction as operator—is the fraction typically conceived as a ratio.

Despite the fact that ratios and fractions do not share identical meanings, many ratios can be meaningfully reinterpreted as fractions. Reconsider the 2 to 5 ratio of vinegar to oil in the salad dressing example. You can reinterpret this part-part comparison as a part-whole comparison (i.e., as two-fifths of something). Remember that the ratio 2 : 5 does not indicate the exact amounts of vinegar or oil used in a particular recipe. The dressing could use 2 cups of vinegar and 5 cups of oil, 4 cups of vinegar and 10 cups of oil, 1 cup of vinegar and 2 $^{1}/_{2}$ cups of oil, and so forth. The recipe might also use 6 tablespoons of vinegar and 15 tablespoons of oil, $^{1}/_{2}$ pint of vinegar and 1 $^{1}/_{4}$ pints of oil, and so on. In each of these recipes, $^{2}/_{5}$ also has meaning as a fraction because each recipe calls for two-fifths as much vinegar as oil.

For example, consider a salad dressing recipe that calls for 4 cups of vinegar and 10 cups of oil. The fact that 4 is $^{2}/_{5}$ of 10 can be illustrated visually. Figure 1.19 separates the 10 cups into 5 equal groups, or fifths. One-fifth of 10 cups is 2 cups. Figure 1.20 then shows two one-fifths of 10 cups, or 4 cups. The amount of vinegar in this recipe (4 cups) is $2/5$ of the amount of oil (10 cups). In sum, the ratio 2 : 5 can be reinterpreted as the fraction $2/5$, to mean that salad dressing made from this recipe always has $2/5$ as much vinegar as oil, no matter what particular amounts of vinegar and oil some-one uses.

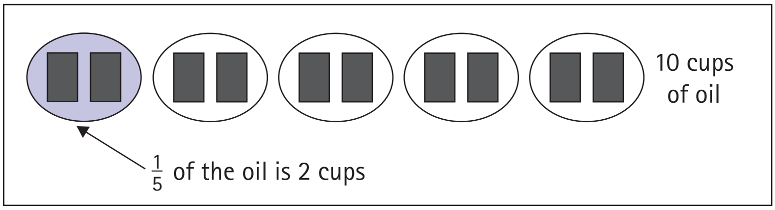

Fig. 1.19. One-fifth of 10 cups

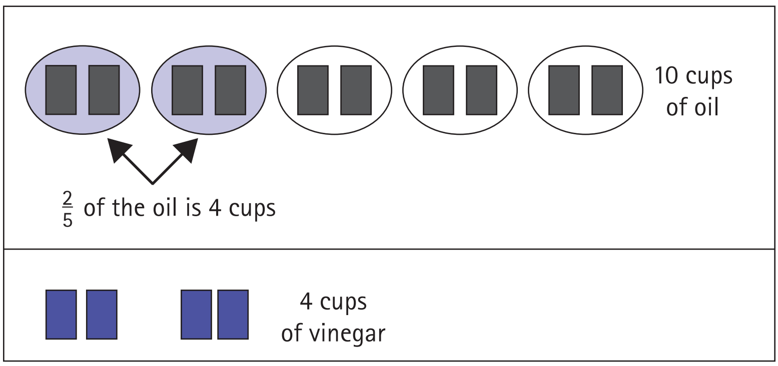

Fig. 1.20. Two-fifths of 10 cups

A second way to interpret the ratio 2 : 5 as the fraction $2/5$ is possible. Suppose that you use 2 cups of vinegar and 5 cups of oil to make the salad dressing. Figure 1.21 shows the "joining" of the vinegar and oil to form a batch of salad dressing. You maintain the ratio 2 : 5 if you partition the batch into 5 equal parts. Your partitioning of the batch partitions both the oil and vinegar into 5 equal parts. Splitting 5 cups of oil into 5 equal parts yields 1 cup of oil in each part. Splitting 2 cups of vinegar into 5 equal parts is more difficult. One way is to split the first cup of vinegar into 5 equal parts, which yields $1/5$ cup of oil in each part. By repeating this process with the second cup, you obtain another $1/5$ cup in each part. Altogether, if you partition 2 cups of oil into 5 equal parts, you have $2/5$ of a cup of oil in each part. Consequently, salad dressing made with $2/5$ cup of vinegar and 1 cup of oil, as illustrated in figure 1.22, maintains the 2 : 5 ratio of vinegar to oil.

In sum, the ratio 2 : 5 (meaning "2 parts vinegar to 5 parts oil") can be reinterpreted as the fraction $2/5$ in two different ways. The first way is to say that in this salad dressing recipe the amount of vinegar is always $2/5$ the amount of oil, no matter what particular amounts of vinegar and oil someone uses to make the dressing.

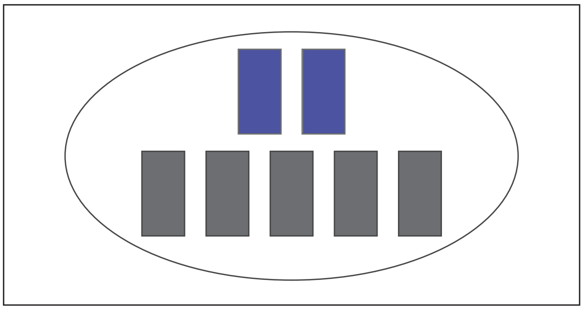

Fig. 1.21. A composed unit of 2 cups of vinegar and 5 cups of oil

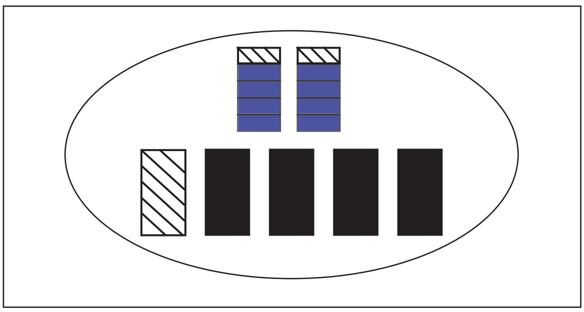

Fig. 1.22. One-fifth of the batch is $2/5$ cup of vinegar and 1 cup of oil

This interpretation is based on understanding the ratio 2 : 5 as a multiplicative comparison—namely, that 2 is $2/5$ of 5. The second way to interpret the ratio as a fraction is to think of the two-fifths as referring to the pairing of $2/5$ cup of vinegar with 1 cup of oil, which maintains the recipe. This interpretation is based on understanding the ratio 2 : 5 as a composed unit, and then partitioning that unit into five equal parts. Reflect 1.4 invites you to apply these two ways of reinterpreting a ratio as a fraction in a different real-world context.

Reflect 1.4

Water is being pumped through a hose into a large swimming pool so that 3 gallons collect in the pool every 4 minutes. What are two different ways to reinterpret the ratio 3 : 4 as the fraction $3/4$ in this context? What are two ways to reinterpret the ratio 4 : 3 as the improper fraction $4/3$?

One way to reinterpret the ratio 3 : 4 (3 gallons every 4 minutes) as the fraction $3/4$ is to say that the number of gallons of water in the pool is always $3/4$ of the number of minutes that have passed, assuming that the water continues to flow into the pool at a constant rate. For example, after 4 minutes, 3 gallons of water are in the pool, and 3 is $^3/_4$ of 4. Similarly, after 20 minutes, 15 gallons of water are in the pool, and 15 is $^3/_4$ of 20. This interpretation is based on thinking of the ratio 3 : 4 as a multiplicative comparison—namely, that 3 is $^3/_4$ of 4. A second way to reinterpret the ratio 3 : 4 as the fraction $^3/_4$ is to consider that $^3/_4$ of a gallon is the amount of water that needs to flow into the pool in 1 minute to maintain the same pumping rate. This interpretation is based on thinking of the ratio 3 : 4 as a composed unit and then partitioning that unit into four equal parts.

The pumping rate can also be captured by the ratio 4 : 3, meaning that 4 minutes elapse for every 3 gallons of water that are pumped into the pool. This ratio can be reinterpreted as the improper fraction $^4/_3$ in two ways. The first way is to say that the number of minutes that elapse is always $^4/_3$ times the number of gallons of water that has flowed into the pool in that time. For example, 12 gallons are pumped in 16 minutes, and $12 \times {}^4/_3 = 16$. This interpretation is based on understanding the ratio 4 : 3 as a multiplicative comparison—namely, that 4 is $^4/_3$ (or $1^1/_3$) times 3. The second way to reinterpret the ratio 4 : 3 as the improper fraction $^4/_3$ is to consider that $^4/_3$ minutes is the amount of time that it takes to pump 1 gallon of water into the pool. This interpretation is based on joining 4 minutes and 3 gallons into a composed unit and partitioning that unit into three equal parts.