Essential Understanding Series

Developing

# Essential Understanding *of* Ratios, Proportions, *and* Proportional Reasoning *for* Teaching Mathematics *in* Grades 6–8

Joanne Lobato
San Diego State University
San Diego, California

Amy B. Ellis
University of Wisconsin–Madison
Madison, Wisconsin

Randall I. Charles
*Volume Editor*
Carmel, California

Rose Mary Zbiek
*Series Editor*
The Pennsylvania State University
University Park, Pennsylvania

NCTM
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

# Essential Understanding 7

*Proportional reasoning is complex and involves understanding that—*
- *equivalent ratios can be created by iterating and/or partitioning a composed unit;*
- *if one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and*
- *the two types of ratios—composed units and multiplicative comparisons—are related.*

The idea of forming a ratio as a composed unit is a foundational concept that is not, by itself, indicative of sophisticated ratio reasoning. In fact, some researchers refer to the formation of a composed unit as *pre-ratio* reasoning (Lesh, Post, and Behr 1988). Essential Understanding 7 presents three crucial aspects of sophisticated proportional reasoning. These three components are presented in order of increasing sophistication, although not everyone comes to an understanding of them in this particular order. The discussion that follows is an introduction to these ideas; a full development of them is beyond the scope of this book.

## Creating equivalent ratios

At the beginning levels of proportional reasoning, students iterate (repeat) and/or partition (break into equal-sized parts) a composed unit to create a family of equivalent ratios. For example, consider the following problem:

> Begin with a ramp that is 3 centimeters high and has a base that is 4 centimeters long. Make all the ramps you can that have the same steepness as the original ramp but are not identical to it.

If a student makes a copy of the original ramp, then both ramps have the same steepness, since neither the height nor the length of the base changed (see fig. 1.26). Aligning the ramp and its copy "tip to tip," as shown in figure 1.27, will not change the steepness of either ramp. The resulting ramp, with a height of 6 centimeters

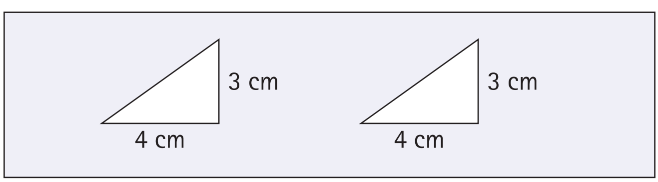


**Fig. 1.26.** A ramp and an identical copy of it

and a base of 8 centimeters, has the same steepness as the original ramp. The iteration process can be continued to create other ramps with the same steepness: a ramp with a height of 12 centimeters and a base of 16 centimeters, one with a height of 21 centimeters and a base of 28 centimeters, and so forth.

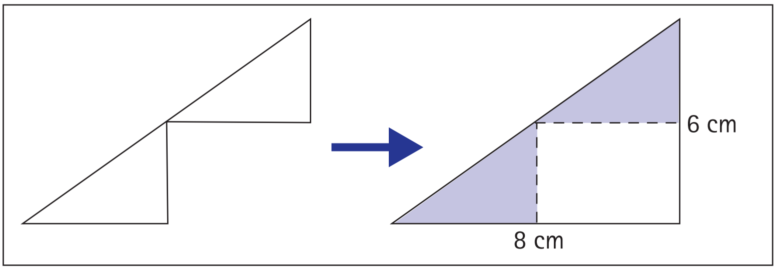


**Fig. 1.27.** A new ramp with the same steepness as the original, made by aligning the original and its copy tip to tip (dotted lines complete the drawing of the new ramp)

Students can also partition the original ramp to form new ramps of equal steepness. Partitioning the height of the original ramp into two equal parts and partitioning the base into two equal parts results in a new ramp with a height of $1^1/_2$ centimeters and a base of 2 centimeters (see fig. 1.28). Students can verify that the new ramp has the same steepness as the original ramp by iterating the new ramp and stacking as before to obtain the original ramp. They can use partitioning to create additional ramps with the same steepness. For example, partitioning the height and base of the original ramp into thirds results in a new ramp with a height of 1 centimeter and a base of $1^1/_3$ centimeters.

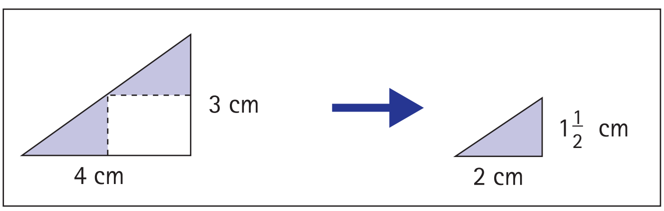


**Fig. 1.28.** Partitioning a ramp to form a new ramp with the same steepness

Students can combine iterating and partitioning. For example, suppose students are asked to determine the height of a new ramp with a base of 5 centimeters and the same steepness as the original 3 : 4 ramp. This is a much harder problem for students because of the relatively small difference between 4 and 5 centimeters. However, they can combine partitioning and iterating to tackle this problem.

Consider the thinking of one middle school student, Marco. He realized that the base of the new ramp was 1 centimeter more than the base of the original ramp, so he decided to find the height of a ramp that had a base of 1 centimeter and the same steepness as the original ramp. He partitioned the 3 : 4 original ramp into 4 equal parts to obtain a $3/4$ : 1 ramp. He then iterated and stacked the $3/4$ : 1 ramp five times so that the base of the new ramp was 5 centimeters. The height new ramp was $15/4$, or $3\,3/4$, centimeters, since it contained five ramps, each with a height of $3/4$ centimeter.

## Maintaining a proportional relationship

An important part of developing more sophisticated proportional reasoning is the ability to truncate the work of iterating a composed unit by using the arithmetic operation of multiplication. To accomplish this, students need to move from simply repeating a composed unit multiple times until they reach a particular goal to being able to anticipate the number of groups that they need.

Consider the work of a middle school student, Andrea. She needed to determine the base of a ramp with a height of 27 centimeters and the same steepness as the original ramp—again, the ramp with a height of 3 centimeters and a base of 4 centimeters. Andrea began by drawing a picture of four stacked ramps like the original. She determined the height of the resulting new ramp by adding the heights of the stacked ramps (3 + 3 + 3 + 3 = 12 centimeters). Andrea realized that she had not used enough copies of the original ramp. She then added another to the stack and again added to determine the height of the new ramp (3 + 3 + 3 + 3 + 3 = 15 centimeters). Andrea continued this process until she drew a new ramp with a height of 27 centimeters and a base of 36 centimeters. Although she eventually arrived at a correct response, her reasoning had not achieved the sophistication demonstrated by David's response, discussed below.

David approached the problem by imagining the height of the new ramp (27 centimeters) as made up of 9 groups of 3 centimeters. As a result, David could conceive of multiplying the height of the original ramp by 9 (3 centimeters × 9 = 27 centimeters). Because David recognized that he needed to iterate the entire 3 : 4 ramp 9 times, he knew that he should also multiply the base by 9 (4 centimeters × 9 = 36 centimeters).

David's work is consistent with understanding that multiplication can abbreviate the longer process of repeated iteration. For students like Andrea, a critical part of developing this understanding is to have repeated experiences that prompt them to reflect on the number of groups that they have formed as a result of iterating. For example, Andrea was able to solve the problem through repeated iteration and counting. She may have been unaware that she used nine copies of the original ramp to create the new ramp. Asking students to reflect on the number of groups (in this instance, ramps) that they used is a critical part of their eventually becoming able to anticipate the number of groups that they need.

It is possible to generalize the understanding reflected in David's work: If one quantity is multiplied by a particular factor, then the other quantity must also be multiplied by the same factor to maintain the proportional relationship. Similarly, if one quantity is divided by a factor, then the other quantity must be divided by the same factor to maintain the proportional relationship. To achieve this understanding, students need to link the act of partitioning to the operation of division and develop an awareness of the number of parts that they create when they partition repeatedly.

For example, suppose that a student needs to determine the base of a ramp that has a height of 1 centimeter and the same steepness as the original 3 : 4 ramp. By imagining 3 centimeters as composed of 3 groups of 1 centimeter, the student can conceive of partitioning the height of the original ramp into 3 equal groups. By linking partitioning with the arithmetic operation of division, the student can obtain the desired height by dividing 3 centimeters by 3 to get 1 centimeter. Because the height is divided by 3, the base must also be divided by 3, and 4 centimeters ÷ 3 is $^4/_3$, or $1^1/_3$, centimeters. Thus, a ramp with a height of 1 centimeter and a base of $1^1/_3$ centimeters will have the same steepness as the ramp with a height of 3 centimeters and a base of 4 centimeters. Dividing each quantity (the height and the base) by the same factor, 3, can also be thought of as multiplying each quantity by a factor of $^1/_3$. In fact, understanding that maintaining a proportional relationship involves multiplying each quantity by the same factor can be extended to include fractional factors.

Reconsider Marco's reasoning, presented previously. Marco needed to find the height of a ramp with a base of 5 centimeters and the same steepness as the original 3 : 4 ramp. Marco partitioned the 3 : 4 ramp into 4 equal parts to obtain a $^3/_4$ : 1 ramp. He then iterated and stacked the $^3/_4$ : 1 ramp five times so that the base of the new ramp was 5 centimeters. As a result, the height of the new ramp was $^{15}/_4$, or $3^3/_4$, centimeters, since it contained five ramps, each with a height of $^3/_4$ centimeters.

Eventually, Marco should develop his thinking to understand the use of fractional factors in such a context. For example, he could begin by realizing that 5 centimeters (the base of the new ramp) is $^5/_4$ of 4 centimeters (the base of the original ramp). Identifying the factor $^5/_4$ can grow out of Marco's reflection on his use of iterating and partitioning. Marco found $^1/_4$ of 4 centimeters by partitioning 4 centimeters into 4 equal parts. Conceptually, this work is the same as finding $^1/_4 \times 4$ centimeters. Then Marco iterated the result 5 times. This activity is the same conceptually as taking 5 one-fourths of 4 centimeters, which is $^5/_4 \times 4$ centimeters.

Once a student realizes that 5 centimeters is $5/4 \times 4$ centimeters, he or she can complete the problem by finding $5/4 \times 3$ centimeters, which is $15/4$, or $3\,3/4$, centimeters. In sum, students like Marco are close to realizing that they can maintain a proportional relationship by multiplying each quantity by the same factor $a/b$.

## Relating the two types of ratios

The discussion thus far has focused on proportional reasoning strategies that rely on thinking of ratios as composed units, because this is usually the easier entry point for middle school students. However, it is also important that students learn to work with multiplicative comparisons and connect these two types of ratios. For example, consider the set of heights and lengths (bases) of all of the equally steep ramps that have been discussed so far in relation to Essential Understanding 7 (see fig. 1.29).

| Height (cm) | Length of Base (cm) |
|---|---|
| 3 | 4 |
| 6 | 8 |
| 9 | 12 |
| 12 | 16 |
| 21 | 28 |
| 27 | 36 |
| 1.5 | 2 |
| $3/4$ | 1 |
| 1 | $4/3$ |
| $3\,3/4$ | 5 |

**Fig. 1.29. Heights and lengths (bases) for a set of equally steep ramps (with shaded rows showing the unit ratios)**

In each case, the height is $3/4$ of the length of the base, and the length of the base is $4/3$ times the height. These two ratios, $3/4$ and $4/3$, are multiplicative comparisons. To form the ratio $3/4$, students can ask, "What part of the length of the base is the height?" To form the ratio $4/3$, they can ask, "How many times greater is the length of the base than its height?" Using multiplicative comparisons is a powerful proportional reasoning strategy. For example, to find the length of the base of a ramp that has a height of 16 centimeters and the same steepness as the original 3 : 4 ramp, students can simply multiply the height by $4/3$ (16 centimeters $\times\ 4/3 = 21\,1/3$

centimeters). Similarly, if they have the length of the base of a ramp of this steepness, they can find the height of the ramp simply by multiplying the base by $^3/_4$.

It is important for students to connect composed units with multiplicative comparisons. Perhaps the easiest way for them to see the connection is by looking at either of the unit ratios (shown in the shaded rows in fig. 1.29). Consider the connections made by Manuel, a seventh grader. Manuel formed a composed unit of 1 centimeter (height) and $^4/_3$ centimeter (length of the base). He iterated the $1 : {}^4/_3$ ramp to form other ramps of equal steepness. By iterating $1 : {}^4/_3$ twice, he obtained a ramp with a height of 2 centimeters and a base of $2^2/_3$ centimeters. By iterating $1 : {}^4/_3$ three times, he found a ramp with a height of 3 centimeters and a base of 4 centimeters. When asked for the length of the base of a ramp with a height of 8 centimeters, Manuel reasoned that a height of 8 centimeters was made up of eight groups of 1 centimeter. For each 1 centimeter of height, he needed $^4/_3$ centimeters in the base. Because he needed eight groups of $^4/_3$ centimeters for the base, he multiplied $8 \times {}^4/_3$. Manuel went on to find the bases of other ramps of equal steepness by multiplying the given heights by $^4/_3$.

Manuel appeared to understand that the base of each of these equally steep ramps was $^4/_3$ times as great as its height. This suggests that he formed a multiplicative comparison between the bases and heights of the ramps by expanding on his initial use of composed units. This connection between multiplicative comparisons and composed units allowed Manuel to write an equation to represent the relationship between the height and base of any ramp in this “same steepness” family. Specifically, Manuel wrote $H \times {}^4/_3 = L$ and explained that could find any length ($L$) of the base of any ramp by multiplying its height ($H$) by $^4/_3$.