Essential Understanding Series

Developing

Essential Understanding *of* Ratios, Proportions, *and* Proportional Reasoning *for* Teaching Mathematics *in* Grades 6–8

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Essential Understanding 2

A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.

There are two ways to form a ratio, both of which involve coordinating two quantities. One way is by comparing two quantities multiplicatively. The second way is by joining or composing the two quantities in a way that preserves a multiplicative relationship.

A ratio as a multiplicative comparison

One way to form a ratio is to create a *multiplicative comparison of two quantities*. For example, consider comparing the lengths of the two worms in figure 1.7. Worm A is 6 inches long, and worm B is 4 inches long. The lengths of the worms can be compared in two ways—additively and multiplicatively. *Additive comparisons* of the lengths would pose and answer questions such as the following:

- How much longer is worm A than worm B? (Worm A is 2 inches longer than worm B.)
- How much shorter is worm B than worm A? (Worm B is 2 inches shorter than worm A.)

By contrast, *multiplicative comparisons* would consider questions like those below:

- How many times longer is worm A than worm B? (Worm A is $1\,^1/_2$ times the length of worm B.)
- The length of worm B is what part, or fraction, of the length of worm A? (Worm B is $^2/_3$ the length of worm A.)

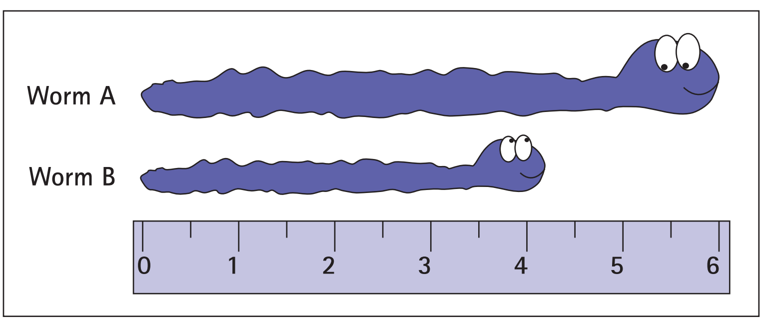

Fig. 1.7. Comparing the lengths of two worms

A multiplicative comparison is a ratio; an additive comparison is not. In general, forming a multiplicative comparison involves asking, "How many times greater is one thing than another?" or "What part or fraction is one thing of another?"

Mathematics uses several conventional notations to represent ratios. You might write the ratio of the length of worm A to the length of worm B as $1\frac{1}{2}:1$, $1\frac{1}{2}$ to 1, or simply $1\frac{1}{2}$. You could also report equivalent ratios, such as 3 : 2, 3 to 2, or $\frac{3}{2}$, as well as 6 : 4, 6 to 4, or $\frac{6}{4}$. You could express the ratio of the lengths of worm B to worm A as 2 : 3, 2 to 3, or $\frac{2}{3}$, as well as 4 : 6, 4 to 6, or $\frac{4}{6}$, in addition to $1:1\frac{1}{2}$ or 1 to $1\frac{1}{2}$.

Reflect 1.3

Suppose that you have made a batch of green paint by mixing 2 cans of blue paint with 7 cans of yellow paint. What are some other combinations of numbers of cans of blue paint and yellow paint that you can mix to make the same shade of green? Solve the problem in two different ways—first by using a multiplicative comparison and then by using a composed unit.

Essential Understanding 7

Proportional reasoning is complex and involves understanding that—

- *equivalent ratios can be created by iterating and/or partitioning a composed unit;*
- *if one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and*
- *the two types of ratios—composed units and multiplicative comparisons—are related.*

You can use a picture like that in figure 1.9 to solve the problem by using a multiplicative comparison. First, determine how many times greater the amount of yellow paint is than the amount of blue paint in the original batch of green paint. One way to do this is to form a group of 2 cans of blue paint. Then find the number of groups of 2 cans that you can make with 7 cans. It takes $3^1/_2$ groups of 2 cans of blue paint to match the amount of yellow paint (see fig. 1.10). Thus, the ratio of yellow paint to blue paint is $3^1/_2$ or 3.5.

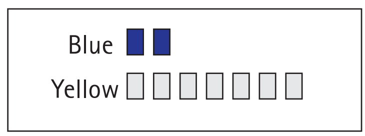

Fig. 1.9. Two cans of blue paint and 7 cans of yellow paint

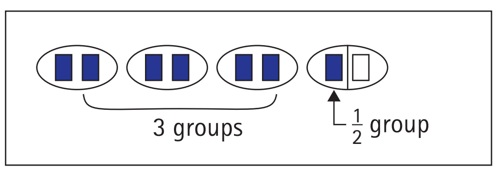

Fig. 1.10. Three-and-a-half groups of 2 cans of blue paint yield an amount equal to the amount of yellow paint in the original batch.

You can use this ratio to find other combinations of blue and yellow paint that result in the same shade of green paint. For example, if you use 5 cans of blue paint, then you need to mix in $3^1/_2$ times as many cans of yellow paint. As shown in figure 1.11, $3^1/_2$ groups of 5 cans are equal to $17^1/_2$ cans. Thus, one way to make a new batch of paint in the same shade of green as the original batch is to use 5 cans of blue paint and $17^1/_2$ cans of yellow paint.

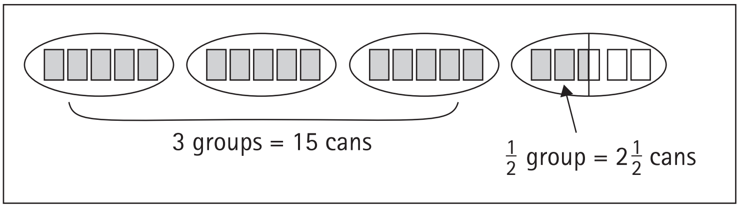

Fig. 1.11. The number of cans of paint in $3\,^1/_2$ groups of 5 cans