Essential Understanding Series

Developing Essential Understanding *of* Ratios, Proportions, *and* Proportional Reasoning *for* Teaching Mathematics *in* Grades 6–8

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A ratio as a composed unit

Another way to form a ratio is by *composing* (joining) two quantities to create a new unit. Evidence of the formation of a *composed unit* often appears in a student's iterating (repeating) or partitioning (breaking apart into equal-sized sections) of a composed unit. Brad's iterating of "10 centimeters in 4 seconds" in the earlier example involving the clown and the frog offers evidence of his formation of a composed unit.

In fact, Brad's discovery led to a flurry of activity, in which other students used the 10 : 4 unit to create new "same speed" values. For example, Denise iterated the 10 : 4 unit three times to conclude that walking 30 centimeters in 12 seconds was the same speed as walking 10 centimeters in 4 seconds. Terry partitioned the 10 : 4 unit into four equal parts, formed a new composed unit of 2.5 : 1 (indicated by the shaded section in fig. 1.8), and then iterated the 2.5 : 1 unit four times to re-create 10 centimeters in 4 seconds. He explained why walking 2.5 centimeters in 1 second was the same speed as walking 10 centimeters in 4 seconds by stating, "It would be like he's walking one-fourth of the 10 and 4; it's like one-fourth of each thing," meaning $^1/_4$ of the 10 centimeters and $^1/_4$ of the 4 seconds.

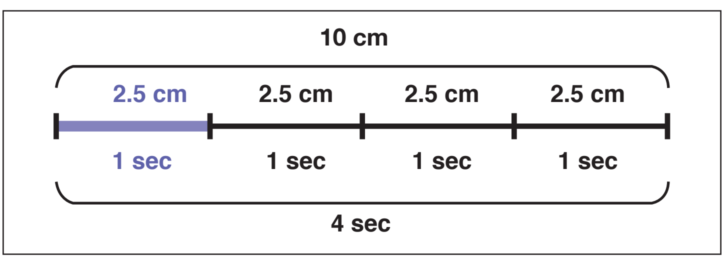

Fig. 1.8. A diagram showing that walking 2.5 centimeters in 1 second is the same speed as walking 10 centimeters in 4 seconds (Lobato and Thanheiser 2002, p. 174)

Forming a ratio as a composed unit does not by itself mean that the student has attained the sophisticated understanding of proportionality that is reflected in the big idea of ratios, proportions, and proportional reasoning. Forming a composed unit is a rudimentary, yet foundational concept, which can be used in conjunction with other essential understandings (especially Essential Understanding 7) to develop an understanding of the big idea of proportional reasoning.

How do the two ways of thinking about a ratio—as a multiplicative comparison and as a composed unit—help in problem solving? Reflect 1.3 encourages you to consider the usefulness of the two concepts of a ratio expressed in Essential Understanding 2.

Reflect 1.3

Suppose that you have made a batch of green paint by mixing 2 cans of blue paint with 7 cans of yellow paint. What are some other combinations of numbers of cans of blue paint and yellow paint that you can mix to make the same shade of green? Solve the problem in two different ways—first by using a multiplicative comparison and then by using a composed unit.

Essential Understanding 7

Proportional reasoning is complex and involves understanding that—

- *equivalent ratios can be created by iterating and/or partitioning a composed unit;*
- *if one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and*
- *the two types of ratios—composed units and multiplicative comparisons—are related.*

You can use a picture like that in figure 1.9 to solve the problem by using a multiplicative comparison. First, determine how many times greater the amount of yellow paint is than the amount of blue paint in the original batch of green paint. One way to do this is to form a group of 2 cans of blue paint. Then find the number of groups of 2 cans that you can make with 7 cans. It takes $3\frac{1}{2}$ groups of 2 cans of blue paint to match the amount of yellow paint (see fig. 1.10). Thus, the ratio of yellow paint to blue paint is $3\frac{1}{2}$ or 3.5.

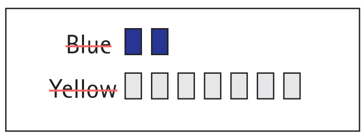

Fig. 1.9. Two cans of blue paint and 7 cans of yellow paint

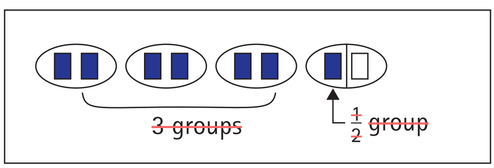

Fig. 1.10. Three-and-a-half groups of 2 cans of blue paint yield an amount equal to the amount of yellow paint in the original batch.

You can use this ratio to find other combinations of blue and yellow paint that result in the same shade of green paint. For example, if you use 5 cans of blue paint, then you need to mix in $3\frac{1}{2}$ times as many cans of yellow paint. As shown in figure 1.11, $3\frac{1}{2}$ groups of 5 cans are equal to $17\frac{1}{2}$ cans. Thus, one way to make a new batch of paint in the same shade of green as the original batch is to use 5 cans of blue paint and $17\frac{1}{2}$ cans of yellow paint.

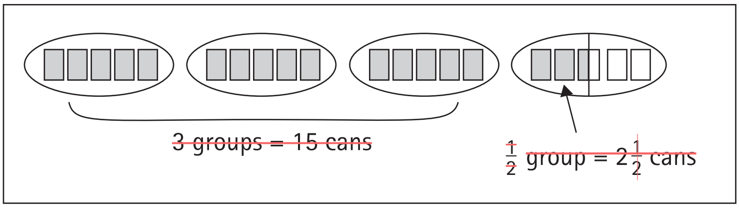

Fig. 1.11. The number of cans of paint in $3\,^1/_2$ groups of 5 cans

You can also solve the paint problem by using a composed unit. First, join 2 cans of blue paint and 7 cans of yellow paint to form a 2 : 7 unit, or batch (see fig. 1.12). Then iterate or partition the 2 : 7 batch to find the number of cans of yellow paint that you need for 5 cans of blue paint. Iterating the 2 : 7 batch twice gives you 4 cans of blue paint and 14 cans of yellow paint. Because you need one more can of blue paint, partition the 2 : 7 batch into two equal parts to obtain 1 can of blue paint and $3\,^1/_2$ cans of yellow paint, as in figure 1.13. In all, $2\,^1/_2$ batches of paint require 5 cans of blue paint and $17\,^1/_2$ cans of yellow paint.

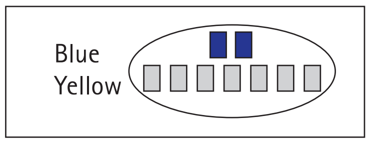

Fig. 1.12. A composed unit of 2 cans of blue paint and 7 cans of yellow paint

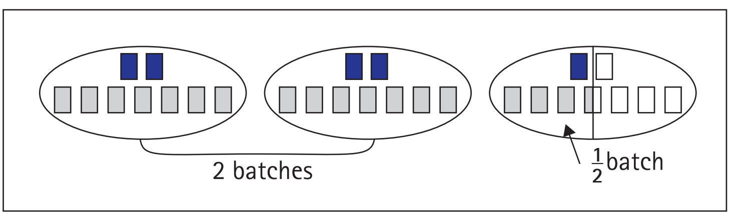

Fig. 1.13. Two-and-a-half batches of green paint, made with 5 cans of blue paint

The notion of a ratio in Essential Understanding 2 as a multiplicative comparison or a composed unit may differ from definitions of ratio in some textbooks. For example, a ratio is commonly defined as a comparison of two quantities. Such a definition is incomplete because it does not clarify whether the comparison is additive or multiplicative.

A ratio is also sometimes defined as a comparison of two numbers that uses division and is often expressed in fraction form. This definition leads students to write expressions such as $^a/_b$ or $a \div b$. However, this definition has several shortcomings. First, it is possible to form a ratio without performing division or creating a fraction. Second, simply by writing "$^a/_b$" or "$a \div b$," a student gives no guarantee that he or she has actually mentally formed a ratio between *a* and *b*.

For instance, consider the middle school task shown in figure 1.14 in the context of making orange juice by mixing cans of orange concentrate with cans of water. Each purple rectangle in the figure represents a can of concentrate, and each white rectangle represents an equal quantity of water. The student must determine the ratio of concentrate to water in the orange juice.

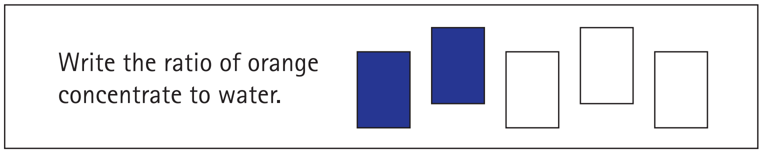

Fig. 1.14. A typical middle school ratio task

Suppose that a student correctly writes "$^{2}/_{3}$" or enters "2 ÷ 3" into a calculator to obtain approximately 0.667. This performance does not mean that he or she understands the situation as involving a ratio; it may simply represent a whole-number counting strategy. The student may count the number of cans of orange concentrate, count the number of cans of water, and write a 2 "over" a 3, without understanding the meaning of $^{2}/_{3}$. In fact, many seventh graders can correctly write $^{2}/_{3}$ in this situation but do not demonstrate an ability to reason with ratios in response to a simple follow-up question:

> Does a batch of orange juice made with 2 cans of orange concentrate and 3 cans of water taste equally orangey, more orangey, or less orangey than a batch made with 4 cans of orange concentrate and 6 cans of water?

Typical responses that indicate an inadequate understanding of ratios include (*a*) "the second batch is more orangey because both numbers are bigger" and (*b*) "the second batch tastes more orangey because you used more orange concentrate." By writing "$^{2}/_{3}$," students can give the illusion that they have a greater understanding of ratio than is actually the case. Remember, *forming a ratio is a cognitive task—not a writing task*. And finally, note that defining a ratio in terms of division places an emphasis on numeric calculations. Essential Understanding 3 highlights an important mathematical idea that does not involve numbers but is related to the formation of ratios in real-world situations.