Essential Understanding Series

Developing Essential Understanding *of* Ratios, Proportions, *and* Proportional Reasoning *for* Teaching Mathematics *in* Grades 6–8

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Essential Understanding 1

Reasoning with ratios involves attending to and coordinating two quantities.

Attending to two quantities is an aspect of reasoning with ratios that mathematically knowledgeable adults understand so implicitly that they often do not recognize its importance until they become aware of its absence in the reasoning of children. Before children are able to reason with ratios, they typically reason with a single quantity. This type of reasoning is called *univariate reasoning*. Harel and colleagues (1994) offer an example of this reasoning. Sixth-grade students were shown a picture of a carton of orange juice and were told that the juice was made from orange concentrate and water. Next to the carton in the picture were two glasses—a large glass and a small glass—both filled with orange juice from the carton. The sixth graders were asked if they thought that the orange juice from the two glasses would taste equally orangey, or if they thought that the juice in one glass would taste more orangey than the juice in the other.

The results are fascinating. Half the class responded incorrectly that the juice from the two glasses would not be equally orangey. About half of these students said that the juice in the large glass would taste more orangey, and about half chose the small glass as likely to taste more orangey. Their explanations suggest that they either focused on one quantity—the water or the orange concentrate—or attended to both quantities but did not coordinate them. For example, one student explained that the juice in the large glass would taste more orangey "because the glass is bigger, so it would hold more orange" (p. 333). Other students explained that the juice in the small glass would taste more orangey because a smaller volume would allow less water to get in, which would leave more room for the orange concentrate.

The importance of coordinating two quantities becomes clear in the following example, which shows the intellectual achievement that such coordination can represent for children. In a study by Lobato and Thanheiser (2002), students in a class viewed a computer screen with SimCalc Mathworlds software showing two characters—a clown and a frog—capable of being set to walk at constant speeds. The clown was set to walk 10 centimeters in 4 seconds. The children were asked to enter distance and time values for the frog so that it would walk at the same speed as the clown (see fig. 1.5). The simulation software would then show the two journeys simultaneously, thus providing feedback that students could use to determine whether the values that they entered were correct. This activity presented a challenge for the students. Many used a guess-and-check strategy; for example, one student tried 15 centimeters and 8 seconds and then kept adjusting the time until he arrived at 15 centimeters in 6 seconds. Other students used numeric patterns—for example, doubling the 10 and the 4 to obtain 20 centimeters in 8 seconds.

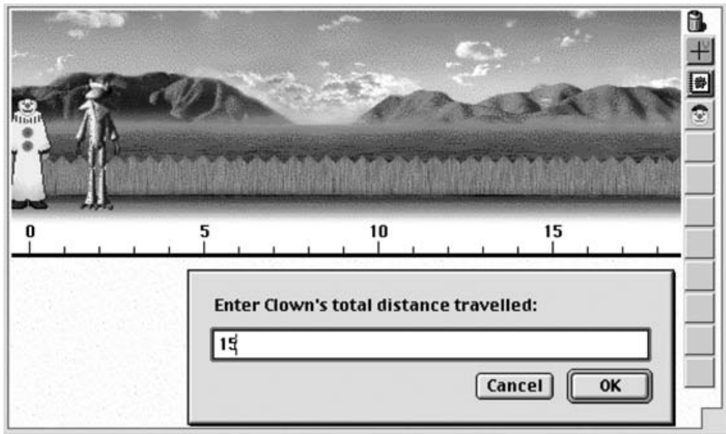

Fig. 1.5. A screen from Roschelle and Kaput's (1996) SimCalc Mathworlds

When the teacher asked the students to explain why walking 20 centimeters in 8 seconds is the same speed as walking 10 centimeters in 4 seconds, one student, Terry, created a drawing that suggests that he had not formed a ratio. Figure 1.6 shows a re-creation of his diagram. He drew lines to represent the distances walked by the two characters without attempting to show that the frog's distance was double the clown's distance. He then relied on calculations, stating, "If you want frog's distance to be 20, then you have to multiply 10 by 2 to get 20. Since you multiplied 10 by 2, you also need to multiply 4 by 2 to get 8." Terry did not explain *why* the time and distance had to be doubled or how multiplying by two could be represented in his drawing.

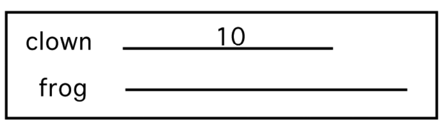

Fig. 1.6. A re-creation of Terry's diagram

Jim, the next student to go to the board, offered a limited explanation that was nearly identical to Terry's. The discussion appeared to stall, when suddenly another student—Brad—had a new idea that he seemed eager to share. Brad explained that doubling works as follows:

> Because the clown is walking the same distance; it's just that he's walking the distance twice… he's walking it once, going li, li, li, li, li, li, [Brad made a "li" sound, evidently to represent time, while his hand retraced the 10 cm line that Terry had drawn], all the way to here [Brad made a vertical hash mark at 10 cm]. Four seconds. Okay. He's going to walk it again. Another four seconds, li, li, li, li, li, li, li, li. Another ten centimeters in four seconds. He's done. (Lobato and Thanheiser 2002, p. 173)

Brad's explanation involved three elements lacking in both Terry's and Jim's work. First, Brad appeared to coordinate time and distance by using sound to represent time while using a hand gesture to represent distance. Second, Brad seemed to coordinate distance and time by forming a "10 centimeters in 4 seconds chunk," which he could repeat. In contrast, Terry seemed to pick one quantity—namely, 20 centimeters—and then produced the other related quantity of 8 seconds. Finally, Brad's image accounted for the frog after the initial 10 centimeters in 4 seconds by noting that the frog walks another 10 centimeters in 4 seconds. By repeating the action of walking 10 centimeters in 4 seconds, the frog will not go faster or slower but will walk at the same speed in both journeys, as well as in the combined journey. In contrast, Terry's explanation did not account for how far the frog walked and in what time after the clown had stopped.

As necessary as it is for students to coordinate two quantities in their reasoning, doing so is not sufficient for understanding ratios. For example, it is possible for students to coordinate two quantities by engaging in a form of reasoning that is different from ratio reasoning—namely, *additive reasoning*. Consider the following situation:

> Jonathan has walked 5 feet in 4 seconds. How long should Rafael take to walk 15 feet if he walks at the same speed as Jonathan?

A seventh grader, Miriam, responded that Rafael should take 14 seconds. She reasoned that 15 feet is 10 more than 5 feet, so you should add 10 seconds to 4 seconds to get 14 seconds. Miriam accounted for both time and distance, but her reasoning was additive because it focused on questions related to "how much more" or "how much less" one quantity is than another. Miriam's work raises the question of what it means to form a ratio.

Essential Understanding 2

A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.

There are two ways to form a ratio, both of which involve coordinating two quantities. One way is by comparing two quantities multiplicatively. The second way is by joining or composing the two quantities in a way that preserves a multiplicative relationship.

A ratio as a multiplicative comparison

One way to form a ratio is to create a *multiplicative comparison of two quantities*. For example, consider comparing the lengths of the two worms in figure 1.7. Worm A is 6 inches long, and worm B is 4 inches long. The lengths of the worms can be compared in two ways—additively and multiplicatively. *Additive comparisons* of the lengths would pose and answer questions such as the following:

- How much longer is worm A than worm B? (Worm A is 2 inches longer than worm B.)
- How much shorter is worm B than worm A? (Worm B is 2 inches shorter than worm A.)

By contrast, *multiplicative comparisons* would consider questions like those below:

- How many times longer is worm A than worm B? (Worm A is $1\frac{1}{2}$ times the length of worm B.)
- The length of worm B is what part, or fraction, of the length of worm A? (Worm B is $\frac{2}{3}$ the length of worm A.)

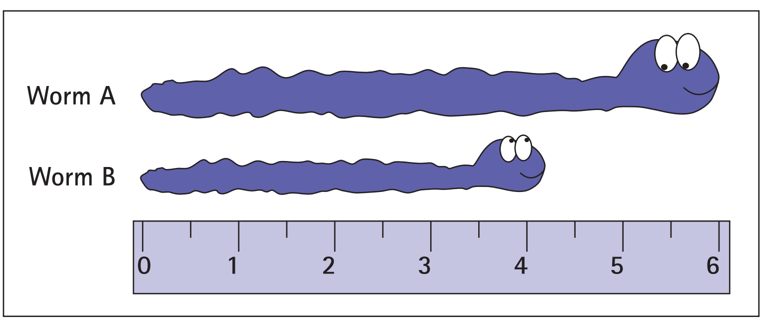

Fig. 1.7. Comparing the lengths of two worms

A multiplicative comparison is a ratio; an additive comparison is not. In general, forming a multiplicative comparison involves asking, "How many times greater is one thing than another?" or "What part or fraction is one thing of another?"

Mathematics uses several conventional notations to represent ratios. You might write the ratio of the length of worm A to the length of worm B as $1\frac{1}{2}:1$, $1\frac{1}{2}$ to 1, or simply $1\frac{1}{2}$. You could also report equivalent ratios, such as $3:2$, 3 to 2, or $3/2$, as well as $6:4$, 6 to 4, or $6/4$. You could express the ratio of the lengths of worm B to worm A as $2:3$, 2 to 3, or $2/3$, as well as $4:6$, 4 to 6, or $4/6$, in addition to $1:1\frac{1}{2}$ or 1 to $1\frac{1}{2}$.