A Series for Teaching Mathematics

# Developing Essential Understanding *of* Ratios, Proportions & Proportional Reasoning

## Grades 6–8

NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Essential Understanding Series

Developing
# Essential Understanding
*of*
# Ratios, Proportions, *and* Proportional Reasoning
*for* Teaching Mathematics *in*
# Grades 6–8

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Chapter 1

# Ratios, Proportions, and Proportional Reasoning:
## The Big Idea and Essential Understandings

A typical instructional unit or chapter on ratio and proportion shows students different ways to write ratios and then introduces a proportion as two equivalent ratios. Next, students usually encounter the cross-multiplication algorithm as a technique for solving a proportion. Does this customary development of ratio and proportion promote a deep understanding of these ideas? Consider an interview with a student named Bonita to think about what it means to reason proportionally.

Bonita was given a problem about a leaky faucet through which 6 ounces of water dripped in 8 minutes. She needed to figure out how much water dripped in 4 minutes. Bonita set up a proportion and used cross multiplication, as shown in figure 1.1, to arrive at a correct response of 3 ounces. Reflect 1.1 invites you to think about Bonita's work on the problem.

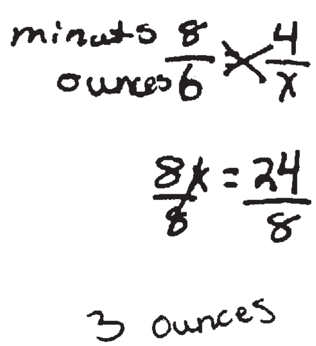


**Fig. 1.1.** Bonita's work on a proportion problem

### Reflect 1.1

Do you think Bonita's work in figure 1.1 shows that she was reasoning proportionally? If so, why do you think so? If not, what do you think she may not have understood?

Bonita's work offers much to like. It is well organized. Bonita labeled the quantities of time and water in her proportion and correctly carried out the cross-multiplication procedure. However, Bonita's responses to three additional tasks suggest that she might not have understood important ideas related to proportional reasoning.

A second task called on Bonita to find the number of ounces that would drip through the same faucet in 40 minutes. To determine whether or not Bonita was procedurally bound to the cross-multiplication method, the interviewer asked her to solve the problem mentally or to use paper and pencil but without applying the algorithm. Bonita was at a loss. She said she couldn't do the problem in her head, and she was unable to do it on paper either. Even after the interviewer changed the specified time from 40 minutes to 16 minutes, Bonita was apparently unable to perform the simple act of doubling mentally or was unaware that doubling would be a reasonable approach.

A third task asked Bonita to solve a problem not posed in the typical form of three numbers given and one missing:

> Crystal placed a bucket under a faucet and collected 6 ounces of water in 20 minutes. Joanne placed a bucket under a second faucet and collected 3 ounces of water in 10 minutes. Were the faucets dripping equally fast or was one dripping faster than the other?

From what you have read so far about Bonita's reasoning, would you expect Bonita to come up with a way to solve this problem? Reflect 1.2 asks you to speculate about Bonita's thinking.

### Reflect 1.2

How do you think Bonita approached the third task set for her by the interviewer? Do you think she was able to reason about it proportionally? Why or why not?

Bonita presented two solutions. First, she said that Crystal's faucet was dripping more slowly than Joanne's because "it took its time." This response suggests that Bonita compared only the amounts of time. Because 20 minutes is greater than 10 minutes, Bonita reasoned that the faucet taking more time was dripping more slowly than the faucet taking less time. Then Bonita changed her mind and said that Crystal's faucet was dripping faster because both amounts—time and water—for Crystal's faucet were greater than the corresponding amounts for Joanne's faucet (i.e., 20 > 10, and 6 > 3).

Bonita's response indicates that she did not form a ratio between the amount of water and the amount of time. In her first solution, she considered only one quantity—elapsed time. In her second attempt, she applied whole-number reasoning to two disconnected pairs of numbers. Bonita's response illustrates the difficulty that many middle school students experience in conceiving that something may remain the same while the values of the two quantities change.

The fourth and final task presented Bonita with the data shown in figure 1.2. She was told that another girl, Cassandra, had collected the data to see how fast her bathtub faucet was leaking. Cassandra had put a large container under the faucet in the morning and then had checked periodically throughout the day to see how much water was in the container. The interviewer constructed the table with uneven time intervals to approximate actual data collection but provided numbers that readily permitted mental calculations.

| Time | Amount of Water |
|---|---|
| 7:00 a.m. | 2 ounces |
| 8:15 a.m. | 12 ounces |
| 9:45 a.m. | 24 ounces |
| 2:30 p.m. | 62 ounces |
| 5:15 p.m. | 84 ounces |
| 6:00 p.m. | 90 ounces |
| 9:30 p.m. | 118 ounces |

**Fig. 1.2. Data collected from a dripping bathtub faucet**

To help Bonita comprehend the situation before encountering any difficult questions, the interviewer asked her how much water dripped between 7 a.m. and 8:15 a.m. Although this question required only simple subtraction ($12 - 2 = 10$), Bonita inappropriately set up a proportion and attempted to solve for $x$, as shown in figure 1.3. This work strongly suggests that Bonita did not understand when it is appropriate to compare numbers by forming a ratio. In sum, although Bonita could correctly execute the proportion algorithm on the first task, her work on the next three tasks demonstrates her poor conceptual understanding.

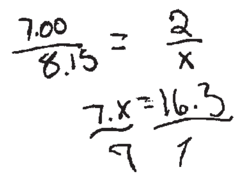

**Fig. 1.3. Bonita's work on the bathtub task**

If Bonita had understood the ideas behind her work, then she should have been able to reason about the faucet that drips 6 ounces of water in 8 minutes by using at least one of the two following methods.

## Proportional Reasoning Method 1

Bonita might have used the method described below to reason about the faucet that dripped 6 ounces in 8 minutes:

- Form a ratio by joining 6 ounces and 8 minutes into a single unit: *6 ounces in 8 minutes.*
- Iterate (repeat) this unit by reasoning that if the faucet drips another 6 ounces in 8 minutes, it does not speed up or slow down since the amounts of time and water are identical. Thus, a faucet that drips 12 ounces in 16 minutes drips at the same rate as one that drips 6 ounces in 8 minutes.
- Similarly, partition, or split, the "6 ounces in 8 minutes" unit in half. A faucet that drips 3 ounces in 4 minutes drips at the same rate as one that drips 6 ounces in 8 minutes.
- Make more challenging partitions. To determine the amount of water that drips in 1 minute, split the unit into eighths by finding $\frac{1}{8}$ of 6 ounces, which is $\frac{6}{8}$, or $\frac{3}{4}$, ounce, and by finding $\frac{1}{8}$ of 8 minutes, which is 1 minute. Thus, a faucet that drips $\frac{3}{4}$ ounce in 1 minute drips at the same rate as one that drips 6 ounces in 8 minutes.
- Combine the actions of iterating and partitioning. For example, quadruple the "6 ounces in 8 minutes" unit to obtain 24 ounces in 32 minutes. Also partition the "6 ounces in 8 minutes" unit into thirds by finding $\frac{1}{3}$ of 6 ounces, which is 2 ounces, and by finding $\frac{1}{3}$ of 8 minutes, which is $\frac{8}{3}$, or $2\frac{2}{3}$ minutes. Combine these results to obtain 26 ounces in $34\frac{2}{3}$ minutes, which is $4\frac{1}{3}$ times the "6 ounces in 8 minutes" unit.

In this manner, construct a large collection of ratios, all of which represent the same dripping rate: 6 ounces in 8 minutes, 12 ounces in 16 minutes, 3 ounces in 4 minutes, $^{3}/_{4}$ ounce in 1 minute, 26 ounces in $34^{2}/_{3}$ minutes, and so on.

## Proportional Reasoning Method 2

Alternatively, Bonita might have reasoned about the faucet dripping 6 ounces of water in 8 minutes by using the following method:

- Compare the two numerical values 6 and 8 (from 6 ounces in 8 minutes) by finding how many times greater 8 is than 6. Eight is $1^{1}/_{3}$ times greater than 6.
- To determine the amount of time that it takes for any amount of water to drip, multiply the value of the water amount by $1^{1}/_{3}$. For example, for 3 ounces of water, it will take $3 \times 1^{1}/_{3}$, or 4, minutes. For 12 ounces, it will take $12 \times 1^{1}/_{3}$, or 16, minutes.
- Construct a collection of ratios by maintaining the factor of $1^{1}/_{3}$. That is, the water amount is always $1^{1}/_{3}$ times greater than the time amount.
- Also compare the values 6 and 8 by finding what fraction 6 is of 8. Six is $^{6}/_{8}$, or $^{3}/_{4}$, of 8.
- To determine the amount of water that drips for any amount of time, multiply the time amount by $^{3}/_{4}$. For example, in 16 minutes, $16 \times {^{3}/_{4}}$, or 12, ounces, of water will drip. In 4 minutes, $4 \times {^{3}/_{4}}$, or 3, ounces of water will drip.

## One Big Idea and Multiple Essential Understandings

The two methods that Bonita could have used to reason proportionally about the faucet dripping 6 ounces in 8 minutes suggest the following big idea related to ratios, proportions, and proportional reasoning: When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor.

*When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor.*

In the situation of the dripping faucet, the water and time values change; yet, infinitely many water and time pairs represent the same dripping rate (e.g., 6 ounces in 8 minutes, 9 ounces in 12 minutes, 3 ounces in 4 minutes, $^{3}/_{4}$ ounce in 1 minute). Any pair in the collection of water and time pairs can be obtained by iterating and/or partitioning any other pair. For example, 9 ounces in 12 minutes is $1^{1}/_{2}$ groups of 6 ounces in 8 minutes and is equal to 3 groups of 3 ounces in 4 minutes. Furthermore, the ratio of time to water in each pair is constant: the number of minutes is $1^{1}/_{3}$ times the number of ounces. The ratio of water to time is also constant: the number of ounces is $^{3}/_{4}$ the number of minutes.

Although the big idea of proportionality may at first seem straightforward, developing an understanding of it is a complex process for students. It involves grasping many essential understandings: