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Productive Struggle

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Editors: Dana Olanoff, Kim Johnson, & Sandy Spitzer

MAKING SENSE OF NON-INTEGER EXPONENTS USING A NUMBER LINE MODEL

John Gruver
Michigan Technological University
jgruver@mtu.edu

Mike Foster
San Diego State University
mikefoster2793@gmail.com

Elizabeth Keysor
Michigan Technological University
egkeysor@mtu.edu

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One area of difficulty for students when reasoning about exponential expressions is correctly manipulating and making sense of exponents (Berezovski, 2004; Cangelosi, et al., 2013; Gol Tabaghi, 2007). Common curricular approaches develop the idea of an exponent as the number of times a number is multiplied by itself (Ellis et al., 2015). However, a central limitation of this "number of factors" meaning for exponents is an inability to make sense of non-integer exponents. While progress has been made in addressing this concern through the expansion of approaches to developing meaning for exponents (Thompson, 2008; Ellis et al., 2015; Kuper and Carlson, 2020), questions remain about how to engender scaling-continuous covariational reasoning (Ellis et al., 2020) to supports students in calculating non-integer exponents. While in scaling-continuous covariation students think about change as it happens over an interval of a fixed size, they can also continuously resize the intervals, a process called zooming. We argue that exponentially scaled number lines can support students in applying scaling-continuous covariational reasoning about non-integer exponents.

An exponentially scaled number line is a number line where same-sized segments of the line represent an increase by the same multiplicative factor. For example, if students were asked to model the growth of bacteria whose amount triples each hour, students might create equally spaced tick marks on a number line labeled 1, 3, 9, etc., on one side of the number line to represent the number of bacteria and 0, 1, 2, etc., on the other side to represent the elapsed time. With support, students could eventually come to realize any same-sized segments of the line represent an increase by the same multiplicative factor of the bacteria. They could also come to realize that on one side of the number line there are exponents, while on the other there are the corresponding powers of three. Students could then make sense of expressions such as $3^{1/2}$ by leveraging the idea that this represents the number of bacteria after half of an hour and will be represented on the number line by a segment that is half as long as the whole hour segment. They could then reason that over the two half-hour segments the number of bacteria grew by the same factor, which means they need the number that when multiplied by itself gives 3, namely $\sqrt{3}$.

We see this model as productive because we believe that it fosters scaling-continuous reasoning. An exponential number line is consistent with representing growth that is continuous and is also consistent with resizing chunks continuously. As students use the number line to explore values between their chunks they will need to reason simultaneously with the change in time and the change in the number of bacteria. Analogous to the linear reasoning behind positioning day 0.5 at the midpoint between of hour 0 and hour 1, the multiplicative reasoning of the number line directs students to find a multiplicative value for a half hour period growth such that two half hour growths results in a one hour growth. This process can then be repeated and the same reasoning applied for successively smaller segments of a growth. We believe this allows for both zooming in on the number line and a continuous image of the exponential function, vis-à-vis the number line, to emerge.

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